OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4764

Mechanics 4

Wednesday **21 JUNE 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- The acceleration due to gravity is denoted by $g\ \text{m s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 4 printed pages.

[Turn over

Section A (24 marks)

1 A spherical raindrop falls through a stationary cloud. Water condenses on the raindrop and it gains mass at a rate proportional to its surface area. At time t the radius of the raindrop is r. Initially the raindrop is at rest and $r = r_0$. The density of the water is ρ.

(i) Show that $\frac{\mathrm{d}r}{\mathrm{d}t} = k$, where k is a constant. Hence find the mass of the raindrop in terms of r_0, ρ, k and t. [6]

(ii) Assuming that air resistance is negligible, find the velocity of the raindrop in terms of r_0, k and t. [6]

2 A rigid circular hoop of radius a is fixed in a vertical plane. At the highest point of the hoop there is a small smooth pulley, P. A light inextensible string AB of length $\frac{5}{2}a$ is passed over the pulley.

A particle of mass m is attached to the string at B. PB is vertical and angle APB $= \theta$. A small smooth ring of mass m is threaded onto the hoop and attached to the string at A. This situation is shown in Fig. 2.

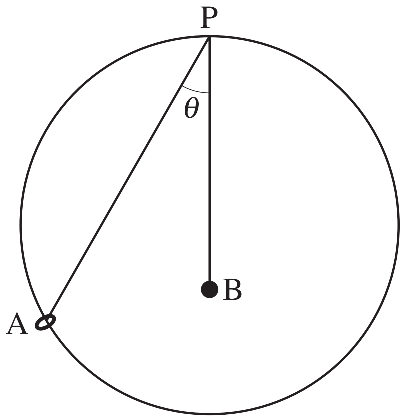


Fig. 2

(i) Show that PB $= \frac{5}{2}a - 2a\cos\theta$ and hence show that the potential energy of the system relative to P is $V = -mga(2\cos^2\theta - 2\cos\theta + \frac{5}{2})$. [4]

(ii) Hence find the positions of equilibrium and investigate their stability. [8]

Section B (48 marks)

3 An aeroplane is taking off from a runway. It starts from rest. The resultant force in the direction of motion has power, P watts, modelled by

$$P = 0.0004m(10\,000v + v^3),$$

where m kg is the mass of the aeroplane and v m s^{-1} is the velocity at time t seconds. The displacement of the aeroplane from its starting point is x m.

To take off successfully the aeroplane must reach a speed of 80 m s^{-1} before it has travelled 900 m.

(i) Formulate and solve a differential equation for v in terms of x. Hence show that the aeroplane takes off successfully. [8]

(ii) Formulate a differential equation for v in terms of t. Solve the differential equation to show that $v = 100 \tan(0.04t)$. What feature of this result casts doubt on the validity of the model? [7]

(iii) In fact the model is only valid for $0 \leqslant t \leqslant 11$, after which the power remains constant at the value attained at $t = 11$. Will the aeroplane take off successfully? [9]

[Question 4 is printed overleaf.]

4 A flagpole AB of length $2a$ is modelled as a thin rigid rod of variable mass per unit length given by

$$\rho = \frac{M}{8a^2}(5a - x),$$

where x is the distance from A and M is the mass of the flagpole.

(i) Show that the moment of inertia of the flagpole about an axis through A and perpendicular to the flagpole is $\frac{7}{6}Ma^2$. Show also that the centre of mass of the flagpole is at a distance $\frac{11}{12}a$ from A. [8]

The flagpole is hinged to a wall at A and can rotate freely in a vertical plane. A light inextensible rope of length $2\sqrt{2}a$ is attached to the end B and the other end is attached to a point on the wall a distance $2a$ vertically above A, as shown in Fig. 4. The flagpole is initially at rest when lying vertically against the wall, and then is displaced slightly so that it falls to a horizontal position, at which point the rope becomes taut and the flagpole comes to rest.

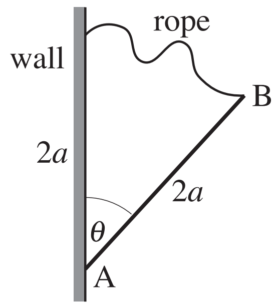


Fig. 4

(ii) Find an expression for the angular velocity of the flagpole when it has turned through an angle θ. [4]

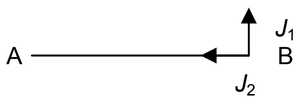
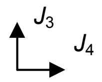
(iii) Show that the vertical component of the impulse in the rope when it becomes taut is $\frac{1}{12}M\sqrt{77ag}$. Hence write down the horizontal component. [5]

(iv) Find the horizontal and vertical components of the impulse that the hinge exerts on the flagpole when the rope becomes taut. Hence find the angle that this impulse makes with the horizontal. [7]

Mark Scheme 4764
June 2006

1(i)	$m = \frac{4}{3}\pi r^3 \rho$	M1	Expression for m	
	$\frac{dm}{dt} = 4\pi r^2 \rho \frac{dr}{dt}$	M1	Relate $\frac{dm}{dt}$ to $\frac{dr}{dt}$	
	$\lambda \cdot 4\pi r^2 = 4\pi r^2 \rho \frac{dr}{dt}$	M1	Use of $\frac{dm}{dt}$ proportional to surface area	
	$\frac{dr}{dt} = \frac{\lambda}{\rho} = k$	E1	Accept alternative symbol for constant if used correctly (here and subsequently)	
	$r = r_0 + kt$	M1	Integrate and use condition	
	$m = \frac{4}{3}\pi\rho(r_0 + kt)^3$	A1		
				6
(ii)	$\frac{d}{dt}(mv) = mg$	M1	N2L	
	$mv = \int mg\ dt = \int \frac{4}{3}\pi\rho(r_0 + kt)^3 g\ dt$	M1	Express mv as an integral	
	$= \frac{4}{3}\pi\rho g\left[\frac{1}{4k}(r_0 + kt)^4 + c\right]$	M1	Integrate	
	$t = 0, v = 0 \Rightarrow c = -\frac{1}{4k}r_0^{\,4}$	M1	Use condition	
	$\frac{4}{3}\pi\rho(r_0 + kt)^3 v = \frac{4}{3}\pi\rho g \cdot \frac{1}{4k}\left[(r_0 + kt)^4 - r_0^{\,4}\right]$	M1	Substitute for m	
	$v = \frac{g}{4k}\left[r_0 + kt - \frac{r_0^{\,4}}{(r_0 + kt)^3}\right]$	A1		
				6
2(i)	$AP = 2a\cos\theta$	M1	Attempt AP in terms of θ	
	$PB = \frac{5}{2}a - 2a\cos\theta$	E1		
	$V = -mg \cdot PB - mg \cdot PA\cos\theta$	M1	Attempt V in terms of θ	
	$= -mg\left(\frac{5}{2}a - 2a\cos\theta\right) - mg\left(2a\cos\theta\right)\cos\theta$			
	$= -mga\left(2\cos^2\theta - 2\cos\theta + \frac{5}{2}\right)$	E1		
				4
(ii)	$\frac{dV}{d\theta} = mga\sin\theta\left(4\cos\theta - 2\right)$	M1	Differentiate	
	$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$	M1	Solve	
	$\Rightarrow \theta = 0$ or $\pm\frac{1}{3}\pi$	A1	For 0 and either of $\frac{1}{3}\pi$ or $-\frac{1}{3}\pi$	
	$\frac{d^2V}{d\theta^2} = mga\sin\theta\left(-4\sin\theta\right) + mga\cos\theta\left(4\cos\theta - 2\right)$	M1 A1	Differentiate again	
	$\theta = 0 \Rightarrow \frac{d^2V}{d\theta^2} = 2mga > 0 \Rightarrow$ stable	M1 F1	Consider sign of V'' in one case Correct deduction for one value of θ	
	$\theta = \pm\frac{1}{3}\pi \Rightarrow \frac{d^2V}{d\theta^2} = -3mga < 0 \Rightarrow$ unstable	F1	Correct deduction for another value of θ N.B. Each F mark is dependent on both M marks. To get both F marks, the two values of θ must be physically possible (i.e. in the first or fourth quadrant) and not be equivalent or symmetrical positions.	
				8

3(i)	$P = Fv = mv\frac{dv}{dx}v$	M1	Use of $P = Fv$	
	$v^2\frac{dv}{dx} = 0.0004(10\,000v + v^3)$	A1	Or equivalent	
	$\int \frac{v}{10\,000 + v^2}\,dv = \int 0.0004\,dx$	M1	Separate variables	
	$\frac{1}{2}\ln\left\|10\,000 + v^2\right\| = 0.0004x + c$	M1	Integrate	
	$v^2 = Ae^{0.0008x} - 10\,000$	M1	Rearrange	
	$x = 0, v = 0 \Rightarrow A = 10\,000$	M1	Use condition	
	$v = 100\sqrt{e^{0.0008x} - 1}$	A1		
	$x = 900 \Rightarrow v = 102.7 > 80$ so successful			
	or $v = 80 \Rightarrow x = 618.37 < 900$ so successful	E1	Show that their *v* implies successful take off	
				8
(ii)	$v\frac{dv}{dt} = 0.0004(10\,000v + v^3)$	F1	Follow previous DE	
	$\int \frac{1}{10\,000 + v^2}\,dv = \int 0.0004\,dt$	M1	Separate variables	
	$\frac{1}{100}\tan^{-1}\left(\frac{1}{100}v\right) = 0.0004t + k$	M1	Integrate	
		A1		
	$t = 0, v = 0 \Rightarrow k = 0$	M1	Use condition	
	$\Rightarrow v = 100\tan(0.04t)$	E1	Clearly shown	
	$v \to \infty$ at finite time suggests model invalid	B1		
				7
(iii)	$t = 11 \Rightarrow v = 47.0781$	B1	At least 3sf	
	Hence maximum $P = 230.049m$	M1	Attempt to calculate maximum *P*	
	$v = 47.0781 \Rightarrow x = 250.237$	M1	Use solution in (i) to calculate *x*	
	$v^2\frac{dv}{dx} = 230.049$	M1	Set up DE for $t \geqslant 11$. Constant acceleration formulae $\Rightarrow$ M0.	
	$\frac{1}{3}v^3 = 230.049x + B$	M1	Separate variables and integrate	
		F1	Follow their maximum *P* (condone no constant)	
	$v = 47.0781, x = 250.237 \Rightarrow B = -22786.3$	M1	Use condition on *x*, *v* (not $v = 0$, not $x = 0$ unless clearly compensated for when making conclusion). Constant acceleration formulae $\Rightarrow$ M0.	
	$v = 80 \Rightarrow x = 840.922$ or $x = 900 \Rightarrow v = 82.0696$	M1	Relevant calculation. Must follow solving a DE.	
	so successful	A1	All correct (accept 2sf or more)	
				9

4(i)	Considering elements of length $\delta x \Rightarrow I = \int_0^{2a} \rho x^2 \, dx$	M1	Set up integral	
	$= \frac{M}{8a^2}\int_0^{2a}\left(5ax^2 - x^3\right)dx$	M1	Substitute for ρ in predominantly correct integral	
	$= \frac{M}{8a^2}\left[\frac{5}{3}ax^3 - \frac{1}{4}x^4\right]_0^{2a}$	M1	Integrate	
	$= \frac{7}{6}Ma^2$	E1		
	Considering elements of length $\delta x \Rightarrow M\bar{x} = \int_0^{2a} \rho x \, dx$	M1	Set up integral	
	$= \frac{M}{8a^2}\int_0^{2a}\left(5ax - x^2\right)dx$	M1	Substitute for ρ in predominantly correct integral	
	$= \frac{M}{8a^2}\left[\frac{5}{2}ax^2 - \frac{1}{3}x^3\right]_0^{2a}$	M1	Integrate	
	$\bar{x} = \frac{11}{12}a$	E1		
				8
(ii)	$\frac{1}{2}I\dot{\theta}^2 = Mg \cdot \frac{11}{12}a(1-\cos\theta)$	M1	KE term in terms of angular velocity	
		B1	$\pm Mg \cdot \frac{11}{12}a\cos\theta$ seen	
		M1	energy equation	
	$\dot{\theta} = \sqrt{\frac{11g}{7a}(1-\cos\theta)}$	A1		
				4
(iii)	 $\theta = \frac{1}{2}\pi \Rightarrow \dot{\theta} = \sqrt{\frac{11g}{7a}}$	F1	Their $\dot{\theta}$ at $\theta = \frac{1}{2}\pi$	
	$2a \cdot (-J_1) = I\left(0 - \sqrt{\frac{11g}{7a}}\right)$	M1	Use of angular momentum	
		A1	Correct equation (their $\dot{\theta}$)	
	$J_1 = \frac{1}{12}M\sqrt{77ag}$	E1		
	$J_2 = \frac{1}{12}M\sqrt{77ag}$	B1	Correct answer or follow their J_1	
				5
(iv)	 $J_4 = J_2 = \frac{1}{12}M\sqrt{77ag}$	M1	Consider horizontal impulses	
		F1	Follow their J_2	
	$J_3 + J_1 = M \cdot \frac{11}{12}a\sqrt{\frac{11g}{7a}}$	M1	Vertical impulse-momentum equation	
		M1	Use of $r\dot{\theta}$	
	$J_3 = \frac{1}{21}M\sqrt{77ag}$	A1	cao	
	$\text{angle } = \tan^{-1}\left(\frac{J_3}{J_4}\right) = \tan^{-1}\left(\frac{\frac{1}{21}M\sqrt{77ag}}{\frac{1}{12}M\sqrt{77ag}}\right)$	M1	Must substitute	
	$= \tan^{-1}\left(\frac{4}{7}\right) \approx 0.519\,\text{rad} \approx 29.7°$	A1	cao (any correct form)	
				7

4764 - Mechanics 4

General Comments

The standard of work varied widely, but most candidates were able to demonstrate some understanding of the principles involved in this unit.

Comments on Individual Questions

1 **Variable mass**

(i) Most candidates were able to show that the rate of change of r was constant, but sometimes made errors with the role of density in their equations. Some candidates made very heavy weather of finding the mass, setting up and solving a differential equation for mass, rather than simply using the formula for volume of a sphere.

(ii) Although some candidates were able to do this part efficiently, many candidates did not seem to realise that they could integrate the Newton's second law equation directly to get mv, but instead expanded $\frac{\mathrm{d}}{\mathrm{d}t}(mv)$ and then set up a differential equation which they were not always able to solve correctly. Some candidates confused their symbols for velocity and volume.

2 **Stability**

(i) This was usually done correctly. Candidates who used diagrams were generally able to make their method clearer than those who relied on words.

(ii) There were many good solutions to this part. Although some errors occurred in differentiation, most errors occurred when solving the resulting equation, with many candidates missing the solution $\frac{1}{3}\pi$ and some introducing solutions representing physically impossible situations. Some candidates mixed up the conditions for stability. Some candidates did not show sufficient working to indicate how they formed their conclusions.

3 **Power and variable acceleration**

(i) This was often done well, but slips in integration were common. Some candidates, after getting their expression for v, did not clearly show that the aeroplane took off successfully.

(ii) This was also often done well, but some omitted the constant of integration. The comments about validity were usually good.

(iii) Good solutions were not common. Many candidates wrongly assumed constant power meant constant acceleration. Many candidates did not use their answer to part (i) to find x after 11 seconds, but either took the longer method of integrating the solution to part (ii) or assumed x to be zero.

4 **Rotation**

This question was found hard; many candidates made little progress beyond part (i).

(i) Most candidates were able to derive the moment of inertia and centre of mass correctly.

(ii) Many did not use energy here and attempted to use Newton's second law but were unable to solve the resulting equation. Of those who did use energy, many made errors in the potential energy term, particularly by not considering the position of the centre of mass.

(iii) There were few good solutions. Most candidates did not use moment of impulse, but attempted to equate impulse to change in angular momentum. Many candidates spent time calculating the angle, whereas the question did specify that the flagpole stops when it is horizontal.

(iv) It is regretted that this part required a technique not explicitly in the specification. Full account was taken of this in the setting of grade boundaries.